Linear representations, symmetric products and the commuting scheme

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Abstract

We show that the ring of multisymmetric functions over a commutative ring is isomorphic to the ring generated by the coefficients of the characteristic polynomial of polynomials in commuting generic matrices. As a consequence we give a surjection from the ring of invariants of several matrices to the ring of multisymmetric functions generalizing a classical result of H.Weyl and F.Junker. We also find a surjection from the ring of invariants over the commuting scheme to the ring of multisymmetric functions. This surjection is an isomophism over a characteristic zero field and induces an isomorphism at the level of reduced structures over an infinite field of positive characteristic.

1 Introduction

Let $\mathbf{K}$ be a commutative ring. For a $\mathbf{K}-$algebra B we denote by $\mathrm{Mat}(n, B)$ be the $B-$module of $n \times n$ matrices with entries in B. We denote by I_n the $n \times n$ identity matrix.

Let $P = \mathbf{K}[y_1, \ldots, y_m]$, $D = \mathbf{K}[x_{ik}]$ and $A = \mathbf{K}[\xi_{kij}]$ be the polynomial rings in variables $\{y_1, \ldots, y_m\}$, $\{x_{ik} : i = 1, \ldots, n,\ k = 1, \ldots, m\}$ and $\{\xi_{kij} : i, j = 1, \ldots, n,\ k = 1, \ldots, m\}$ over the base ring $\mathbf{K}$.

Following C.Procesi [3,10] we introduce the generic matrices. Let $\xi_k = (\xi_{kij})$ be the $n \times n$ matrix whose (i, j) entry is ξ_{kij} for $i, j = 1, \ldots, n$ and $k = 1, \ldots, m$. We call $\xi_1, \ldots, \xi_m$ the generic $n \times n$ matrices. We denote by $A_P =$

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$\mathbf{K}[\xi'_{111},\ldots,\xi'_{m11},\xi'_{112}\ldots,\xi'_{mnn}]$ the residue algebra of A modulo the ideal generated by the relations obtained from the equation $\xi_k\xi_h = \xi_h\xi_k$ for $k,h = 1,\ldots,m$. Here ξ'_{kij} is the class of ξ_{kij} in A_P. We let $\xi'_k = (\xi'_{kij})$ be the $n \times n$ matrix with entries ξ'_{kij} for $i,j = 1,\ldots,n$ and $h = 1,\ldots,m$. We call $\xi'_1,\ldots,\xi'_m$ the generic commuting $n \times n$ matrices.

There is an $n-$dimensional linear representation $\pi_P : P \to \mathrm{Mat}(n, A_P)$ given by mapping y_k to ξ'_k for $k = 1,\ldots,m$ (see [3], §1) The composition $\det\cdot\pi_P$ gives a multiplicative polynomial mapping $P \to A_P$ homogeneous of degree n, (see N.Bourbaki [1] A.IV.54).

We denote by $P^{\otimes n}$ the tensor product n times of P with itself. The symmetric group $\mathfrak{S}_n$ acts on $P^{\otimes n}$ as a group of $\mathbf{K}-$algebra automorphisms by permuting the factors. We denote by $\mathrm{TS}^n P$ the invariants of $P^{\otimes n}$ under $\mathfrak{S}_n$. By N.Roby [13] there is a unique $\mathbf{K}-$algebra homomorphism $\alpha : \mathrm{TS}^n P \to A_P$ such that $\alpha(f(y_1,\ldots,y_m)^{\otimes n}) = \det(\pi_P(f)) = \det(f(\xi'_1,\ldots,\xi'_m))$. Write

$$\det(tI_n - f(\xi'_1,\ldots,\xi'_m)) = t^n + \sum_{k=1}^{n}(-1)^k\psi_k(f)t^{n-k} \qquad (1)$$

to denote the characteristic polynomial of $\pi_P(f) = f(\xi'_1,\ldots,\xi'_m)$. Let C_P be the subalgebra of A_P generated by the coefficients of the characteristic polynomial of $f(\xi'_1,\ldots,\xi'_m)$ for $f \in P$.

We shall prove the following.

Theorem 1 *The map $\alpha : \mathrm{TS}^n P \to A_P$ gives an isomorphism*

$$\mathrm{TS}^n P \cong C_P$$

i.e. the ring of symmetric tensors of order n over a polynomial ring is isomorphic to the ring generated by the coefficients of the characteristic polynomial of polynomials in generic commuting matrices.

The symmetric group $\mathfrak{S}_n$ acts on D by permuting the variables so that for $\sigma \in \mathfrak{S}_n$ we have $\sigma(x_{ik}) = x_{\sigma(i)\,k}$ for all i and k. We denote by $D^{\mathfrak{S}_n}$ the ring of the invariants for this action. It is called the ring of multisymmetric functions, see [15] for a recent reference. There is an obvious $\mathfrak{S}_n-$equivariant isomorphism $D \to P^{\otimes n}$ given by mapping x_{ik} to $1^{\otimes i-1} \otimes y_k \otimes 1^{\otimes n-i}$, therefore we have that $D^{\mathfrak{S}_n} \cong \mathrm{TS}^n P$.

Remark 2 *The proof of Theorem 1 will be based on Th.1 [15] that gives a generating set for the ring of multisymmetric functions over a commutative ring* $\mathbf{K}$. *It can be also proved by using results due to D.Ziplies [17] and F.Junkers [7].*

There is a surjective homomorphism of $\mathbf{K}-$algebra $\Delta : A \to D$ given by

mapping ξ_{kij} to 0 if $i \neq j$ and to x_{ik} otherwise, for $i, j = 1, \ldots, n$ and $k = 1, \ldots, m$. Observe that the (i,i) entry of $\xi_k \xi_h - \xi_h \xi_k$ belongs to $\ker \Delta$ for $k, h = 1, \ldots, m$. Thus Δ factors through a surjective algebra homomorphism $\Delta' : A_P \to D$ defined by mapping ξ'_{kij} to 0 if $i \neq j$ and to x_{ik} otherwise.

The general linear group $\mathrm{G} = \mathrm{GL}(n, \mathbf{K})$ acts on A via the action of simultaneous conjugation on $m-$tuples of matrices $\mathrm{Mat}(n, \mathbf{K})^m$. Namely this action maps ξ_{kij} to the (i,j) entry of $g^{-1}\xi_k g$ for $k = 1, \ldots, m$ and $g \in \mathrm{G}$. Since $\xi_k \xi_h - \xi_h \xi_k = 0$ is invariant by conjugation by elements of G we have that the G acts as a group of automorphism also on A_P. We denote as usual by A_P^{G} the invariants for this action. We will prove the following Theorem.

Theorem 3 *The restriction of Δ to A^{G} is a surjection onto $D^{\mathfrak{S}_n} (\cong \mathrm{TS}^n P)$. The same holds for the restriction of Δ' to A_P^{G}.*

When $\mathbf{K}$ is a characteristic zero field the restriction of Δ' to A_P^{G} gives an isomorphism $A_P^{\mathrm{G}} \cong D^{\mathfrak{S}_n}$.

Let N_P denote the nilradical of A_P. When $\mathbf{K}$ is an infinite field of arbitrary characteristic Δ' induces an isomorphism between $(A_P/N_P)^{\mathrm{G}}$ and $D^{\mathfrak{S}_n}$.

Remark 4 *The proof of Theorem 3 it is based on the observation that $C_P \subset A_P^{\mathrm{G}}$ together with Theorem 1 . Over a characteristic zero field we have that $C_P = A_P^{\mathrm{G}}$ following C.Procesi [10], K.Sibirskiĭ [14], D.Mumford [9] and the result follows.*

In the next section we prove Theorem 1. The third section contains the proof of Theorem 3, some corollaries and remarks.

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2 Determinant and symmetric tensors

We prove Theorem 1.

Let $\mathbf{K}[x_1, \ldots, x_n]$ be the polynomial ring in variables $x_1, \ldots, x_n$ over $\mathbf{K}$. The symmetric group $\mathfrak{S}_n$ acts on $\mathbf{K}[x_1, \ldots, x_n]$ by permuting the variables. The ring $\mathbf{K}[x_1, \ldots, x_n]^{\mathfrak{S}_n}$ of invariants for this action is called the ring of symmetric functions and it is generated by the elementary symmetric functions $e_1, \ldots, e_n$ given by the generating function

$$\prod_{i=1}^{n}(t - x_i) = t^n + \sum_{k=1}^{n}(-1)^k e_k t^{n-k}$$

where the equality is calculated in $\mathbf{K}[t, x_1, \ldots, x_n]$ with t an extra variable (see [8]).

Consider now an element $f \in P$, it gives a homomorphism $\rho_f : \mathbf{K}[x_1, \ldots, x_n] \to D$ by mapping x_i to $f(x_{i1}, \ldots, x_{im})$. We write $e_k(f)$ to denote $\rho_f(e_k)$ for $k = 1, \ldots, n$.

Lemma 5 *$D^{\mathfrak{S}_n}$ is generated by $e_1(f), \ldots, e_n(f)$ with f varying in the set of those monomials in P that are not a proper power of another one.*

PROOF. Th.1[15]. □

Remark 6 *Let δ_k be the $n \times n$ diagonal matrices with $x_{1k}, \ldots, x_{nk}$ on the diagonal, for $k = 1, \ldots, m$. Consider $f(\delta_1, \ldots, \delta_m) \in \mathrm{Mat}(n, D)$, it is a diagonal matrix having diagonal entries $\rho_f(x_1), \ldots, \rho_f(x_n)$. Hence its characteristic polynomial has $i-$th coefficient $(-1)^i e_i(f)$ as observed in [15].*

Lemma 7 *There is a unique algebra homomorphism*

$$\alpha : TS^n P \to C_P$$

such that $\alpha(f \otimes \cdots \otimes f) = \det(f(\xi'_1, \ldots, \xi'_m))$ for all $f \in P$. The homomorphism α is surjective.

PROOF. The homomorphism α is the unique $\mathbf{K}-$algebra homomorphism corresponding to $\det \cdot \pi_P$ which has been introduced in the paragraph just before Theorem 1 . It remains only to prove that the image of α is C_P.

Let $\alpha^* : D^{\mathfrak{S}_n} \to A_P$ be the composition of the isomorphism $D^{\mathfrak{S}_n} \cong TS^n P$ with α. Let t be an extra variable we have the homomorphism

$$\beta = id_{\mathbf{K}[t]} \otimes \alpha^* : \mathbf{K}[t] \otimes D^{\mathfrak{S}_n} \to \mathbf{K}[t] \otimes A_P$$

such that (recall Remark 6)

$$\begin{aligned}\beta(\det(t\otimes 1-1\otimes f(\delta_1,\ldots,\delta_m))) &= \beta(\prod_{i=1}^{n}(t\otimes 1-1\otimes\rho_f(x_i)))\\ &= t^n\otimes 1+\sum_{k=1}^{n}(-1)^k t^{n-k}\otimes\alpha^*(e_k(f))\\ &= t^n\otimes 1+\sum_{k=1}^{n}(-1)^k t^{n-k}\otimes\psi_k(\pi_P(f))\end{aligned}$$

therefore $\alpha(\mathrm{TS}^n P)=C$ by Lemma 5. $\square$

Remark 8 *The generating set recalled in Lemma5 was also known, in essence, to D.Ziplies [17] and F.Junkers [7].*

Lemma 9 *Let $\Delta' : A_P \to D$ be given by mapping ξ'_{kij} to 0 if $i\neq j$ and to x_{ik} otherwise. Then $\Delta'(C_P)=D^{\mathfrak{S}_n}$.*

PROOF. The homomorphism $\Delta'_n : \mathrm{Mat}(n,A_P)\to\mathrm{Mat}(n,D)$ induced by Δ' is such that $\Delta'_n(\xi'_k)=\delta_k$ for $k=1,\ldots,m$. Thus for $f(\xi'_1,\ldots,\xi'_m)\in\pi_P(P)\subset\mathrm{Mat}(n,A_P)$ we have that $\Delta'_n(f)=f(\delta_1,\ldots,\delta_m)$. Hence

$$\begin{aligned}\det(tI_n-\Delta'_n(f)) = \prod_{i=1}^{n}(t-\rho_f(x_i)) &= t^n+\sum_{k=1}^{n}(-1)^k e_k(f)t^{n-k}\\ &= t^n+\sum_{k=1}^{n}(-1)^k\Delta'(\psi_k(\pi_P(f)))t^{n-k}\end{aligned}$$

Thus $\Delta'(C_P)=D^{\mathfrak{S}_n}$ by Lemma 5 and the Lemma follows. $\square$

Proof of Theorem 1 We have $\alpha^*\Delta' = id_{C_P}$ and $\Delta'\alpha^*=id_{D^{\mathfrak{S}_n}}$, thus the result follows thanks to the isomorphism $D^{\mathfrak{S}_n}\cong\mathrm{TS}^n P$. $\square$

3 Invariants

We denote by $M=\mathrm{Mat}(n,\mathbf{K})^m$ the $\mathbf{K}-$module of $m-$tuples of $n\times n$ matrices. The general linear group G acts on M by simultaneous conjugation, so that an element $g\in\mathrm{G}$ maps $(M_1,\ldots,M_m)\in M$ to $(gM_1g^{-1},\ldots,gM_mg^{-1})$. This action induces another on A given by mapping ξ_{kij} to the (i,j) entry of $g^{-1}\xi_k g$ for $i,j=1,\ldots,n$, $k=1,\ldots,m$ and for all $g\in\mathrm{G}$. We denote by A^{G}

the ring of invariants for this action. Let $F = \mathbf{K}\{z_1, \dots, z_m\}$ be the free associative non commutative algebra on $z_1, \dots, z_m$ over the base ring $\mathbf{K}$. There is a linear representation $\pi_F : F \to \mathrm{Mat}(n, A)$ given by mapping z_k to ξ_k for $k = 1, \dots, m$. Consider $\pi_F(f) = f(\xi_1, \dots, \xi_m) \in \mathrm{Mat}(n, A)$ and let t be an extra variable, we write

$$\det(tI_n - f) = t^n + \sum_{i=1}^{n} (-1)^i \theta_i(f) t^{n-i} \tag{2}$$

We denote by C the subring of A generated by the coefficients $\theta_i(f)$ for $f \in F$. It is clear that $C \subset A^{\mathrm{G}}$.

Remark 10 *The previous paragraph is borrowed from [3].*

Consider now the surjective homomorphism $\gamma : A \to A_P$. The kernel of γ is the ideal of A generated by the relations obtained from the equation $\xi_k \xi_h = \xi_h \xi_k$ for $k, h = 1, \dots, m$ and these are invariant. Thus A_P is a $\mathrm{G}-$module and γ is $\mathrm{G}-$equivariant and we have a homomorphism $\gamma : A^{\mathrm{G}} \to A_P^{\mathrm{G}}$. Furthermore one can check that $\gamma(\theta_k(\pi_F(f))) = \psi_k(f(\xi'_1, \dots, \xi'_m))$ for all $f \in F$ and $k = 1, \dots, m$ so that $\gamma(C) = C_P \subset A_P^{\mathrm{G}}$.

The symmetric group $\mathfrak{S}_n$ is embedded in G via the permutation representation. Its image acts on A by restriction of the previous action by permuting the variables. Thus the ring of invariants for this action $A^{\mathfrak{S}_n}$ contains A^{G}. The kernel of $\Delta : A \to D$ is generated by the x_{kij}'s with $i \neq j$, therefore Δ is $\mathfrak{S}_n-$equivariant and we have a homomorphism $\Delta : A^G \to D^{\mathfrak{S}_n}$.

Clearly A_P is an $\mathfrak{S}_n-$module since it is a $\mathrm{G}-$module and the homomorphism γ is $\mathfrak{S}_n-$equivariant because it is $\mathrm{G}-$equivariant. Therefore Δ' is $\mathfrak{S}_n-$equivariant so that $\Delta'(A_P^{\mathrm{G}}) \subset D^{\mathfrak{S}_n}$.

Lemma 11 *Let* $\mathbf{K}$ *be a characteristic zero field. The general linear group* G *is linear reductive thus* $A^{\mathrm{G}} \to A_P^{\mathrm{G}}$ *is surjective.*

PROOF. [9], Chapter 1, pag.26. □

Lemma 12 *Let* $\mathbf{K}$ *be a characteristic zero field. In this case* $C = A^{\mathrm{G}}$ *hence* $A_P^{\mathrm{G}} = C_P$.

PROOF. C.Procesi and K.Sibirskiĭ[10,14] proved that $C = A^{\mathrm{G}}$. The result follows from Lemma 11.

Proof of Theorem 3 Under the induced homomorphisms $\Delta_n : \mathrm{Mat}(n, A) \to \mathrm{Mat}(n, D)$ and $\Delta'_n : \mathrm{Mat}(n, A_P) \to \mathrm{Mat}(n, D)$ we have that $\delta_k = \Delta_n(\xi_k) =$

$\Delta'_n(\xi'_k)$ for $k = 1, \ldots, m$. This implies $\Delta(C) = \Delta'(C_P) = D^{\mathfrak{S}_n}$. Since $C \subset A^{\mathrm{G}}$ and $C_P \subset A_P^{\mathrm{G}}$ we have that $\Delta(A^{\mathrm{G}}) = \Delta'(A_P^{\mathrm{G}}) = D^{\mathfrak{S}_n}$ as claimed in the statement.

Suppose now $\mathbf{K}$ is a characteristic zero field. We have $\Delta' : A_P^{\mathrm{G}} \xrightarrow{\cong} D^{\mathfrak{S}_n}$ by Theorem 1, Lemma 11 and Lemma 12.

Recall that we denoted by N_P the nilradical of A_P, it is a $\mathrm{G}-$module. Hence the action of G on A_P induces another on A_P/N_P so that the natural homomorphism $A_P \to A_P/N_P$ is $\mathrm{G}-$equivariant. The homomorphism Δ' factors through a homomorphism $\Delta'' : A_P/N_P \to D$ because D is reduced. Clearly $\Delta''((A_P/N_P)^{\mathrm{G}}) = \Delta'(A_P^{\mathrm{G}}) = D^{\mathfrak{S}_n}$.

We show now that $\Delta''(A_P/N_P)^{\mathrm{G}} \xrightarrow{\cong} D^{\mathfrak{S}_n}$ when $\mathbf{K}$ is an infinite field of arbitrary characteristic. Let k be the algebraic closure of $\mathbf{K}$. Recall that the rational points of $\mathrm{GL}(n, \mathbf{K})$ are dense in the group $\mathrm{GL}(n, k)$ thus we can suppose $\mathbf{K}$ algebraically closed without any loss of generality (see [11], §6.1). Given a m-tuple $(Z_1, \ldots, Z_m)$ of pairwise commuting matrices there is $g \in \mathrm{G}$ such that $gZ_1\, g^{-1}, \ldots, gZ_m\, g^{-1}$ are all in upper triangular form. Let then $(M_1, \ldots, M_m)$ be a tuple of pairwise commuting matrices in upper triangular form. Consider now a 1-parameter subgroup λ of G. We choose λ such that

$$\lambda(t) = \begin{pmatrix} t^{a_1} & 0 & \ldots & 0 \\ 0 & t^{a_2} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & t^{a_n} \end{pmatrix}$$

for some positive integers $a_1 > a_2 > \cdots > a_n$.
For $i = 1, \ldots, m$ the map

$$\lambda_i : \mathbb{A}^1 \backslash \{0\} \to \mathbb{A}^{n^2},\ t \mapsto \lambda(t)\, M_i\, \lambda(t)^{-1}$$

can be extended to a regular map

$$\overline{\lambda}_i : \mathbb{A}^1 \to \mathbb{A}^{n^2}$$

which sends the origin of $\mathbb{A}^1$ to the diagonal matrix having the same diagonal elements as M_i. It is clear that the latter belongs to the closure of the orbit of M_i for $i = 1, \ldots, n$. Thus we have that in the closure of the orbit of $(M_1, \ldots, M_m)$ there is the (closed) orbit of the m-tuple of diagonal matrices obtained as above. Let now $f \in (A_P/N_P)^{\mathrm{G}}$ be an invariant regular function such that $\Delta''(f) = 0$. Suppose f is not identically zero on the orbits of tuples of commuting matrices, then there is an orbit of a tuple of diagonal matrices over which $f \neq 0$ by continuity i.e. $\Delta''(f) \neq 0$. $\square$

Remark 13 *A classical result due to F.Junker [7] and restated by H.Weyl [16] says that $D^{\mathfrak{S}_n}$ is generated by the restriction of traces to the diagonal matrices i.e. $A^{\mathrm{G}} \to D^{\mathfrak{S}_n}$ is surjective over a characteristic zero field $\mathbf{K}$. Theorem 3 generalizes Junker - Weyl's result to any commutative base ring.*

Corollary 14 *Let $R \cong P/I$ be a $\mathbf{K}-$algebra which is a flat $\mathbf{K}-$module. There are two surjective homomorphisms $A_P^{\mathrm{G}} \to TS^nR$ and $A_P^{\mathrm{G}} \to TS^nR$. are*

PROOF. Since R is flat it is an inverse limit of free $\mathbf{K}-$modules, then by Roby [12] IV, 5. Proposition IV. 5, and Bourbaki [1] Exercise 8(a), AIV. p.89, we have a surjective homomorphism $\mathrm{TS}^nP \to \mathrm{TS}^nR$. The result then follows from Theorem 3. □

3.1 Characteristic zero

From now on $\mathbf{K}$ will be a characteristic zero field.

Corollary 15 *The ideal N_P^{G} is the zero ideal and $A_P^{\mathrm{G}} \cong (A_P/N_P)^{\mathrm{G}}$.*

PROOF. In characteristic zero we have $(A_P/N_P)^{\mathrm{G}} \cong A_P^{\mathrm{G}}/N_P^{\mathrm{G}}$ since G is linear reductive. We also have $A_P^{\mathrm{G}} = C_P$ and $C_P \cong D^{\mathfrak{S}_n} \cong (A_P/N_P)^{\mathrm{G}}$ by Theorem 3. Hence $N_P^{\mathrm{G}} = \{0\}$ and $A_P^{\mathrm{G}} \cong (A_P/N_P)^{\mathrm{G}}$. □

Remark 16 *Corollary 15 implies Theorem 3.3 [4] where $\mathbf{K} = \mathbb{C}$ and was also proved for $\mathbf{K} = \mathbb{C}$ and $m = 2$ by Gan and Ginzburg [6].*

Remark 17 *The affine scheme $\operatorname{Spec} A_P$ is called the commuting scheme. It is conjectured that it is reduced i.e. $N_P = 0$. Corollary 15 gives some support to this conjecture.*

Remark 18 *The ring D^{S_n} is Cohen-Macauley by the Eagon-Hochster theorem[5]. It is also Gorenstein by Lemma 7.1.7 [2]. Then also A_P^{G} is Cohen-Macauley and Gorenstein by Theorem 3.*

Remark 19 *C.Procesi told me that he has independently proved the part of Theorem 3 relative to the characteristic zero case in this way: from H.Weyl [16] one knows that $A^{\mathrm{G}} \to D^{\mathfrak{S}_n}$ is onto. Characteristic zero implies $A^{\mathrm{G}} \to A_P^{\mathrm{G}}$ is onto and its kernel contains the traces of commutators of generic matrices. Theorem 1 and Theorem 2 in [15] jointly say that the kernel of $A^{\mathrm{G}} \to D^{\mathfrak{S}_n}$ is generated by the traces of commutators of generic matrices hence $A_P^{\mathrm{G}} \cong D^{\mathfrak{S}_n}$.*

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